Simplex table – 1

		$C \rightarrow$	20	10	1	0	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	x_{Bi}/y_{ij}
0	s_1	50	3	-3	5	1	0	0	50/3
0	s_2	10	1	0	1	0	1	0	10/1 ←
0	s_3	20	1	-1	4	0	0	1	20/1
	z_j	0	0	0	0	0	0	0	
		$z_j - c_j$	-20	-10	-1	0	0	0	
			↑						

Thus, x_1 should enter and s_2 should leave. Performing the operations $R_1 \leftarrow R_1 - 3R_2$, $R_2 \leftarrow R_2/1$ and $R_3 \leftarrow R_3 - R_2$

Simplex table – 2

		$C \rightarrow$	20	10	1	0	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	x_{Bi}/y_{ij}
0	s_1	20	0	-3	2	1	-3	0	-
20	x_1	10	1	0	1	0	1	0	-
0	s_3	10	0	-1	3	0	-1	1	-
	z_j	30	20	0	20	0	20	0	
		$z_j - c_j$	0	-10	-2	0	20	0	
				↑					

Since $y_{ij} \leq 0$ the given LP has an unbounded solution. In the stately table, both x_3 and x_4 candidates for entering the solution. Because x_2 has most negative net evaluation, it is normally selected as the entering variable. However, all constraints coefficients y_{i2} under x_2 are negative or zero, meaning that x_2 can be increased in differently without violating any of the constraints. Because each unit increase in z by 10, an infinite increase in x_2, will also result in an infinite increase in z. Thus the value of z increases in the direction of x_2

Infeasible Solution:

If the net evaluation $z_j - c_j$ is positive* for all j, and at least one artificial variable present in the basic solution, then the L.P.P. has no feasible solution.

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* if the objective function is to maximize, and negative for minimization problem.

EXERCISES

1. Use charnes penalty to

Maximize $Z = 6x_1 - 3x_2 + 2x_3$

Subject to the constraints

$2x_1 + x_2 + x_3 \le 16$

$3x_1 + 2x_2 + x_3 \le 18$

$x_2 - 2x_3 \ge 8,\ \ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

2. Show that the following LPP has an unbounded solution.

Maximize $Z = 3x_1 - x_2$

Subject to the constraints

$2x_1 + x_2 \ge 2$

$x_1 + 3x_2 \le 2$

$x_2 \le 4$

$x_1 \ge 0, x_2 \ge 0.$

3. Solve the following problem using big M method:

Maximize $Z = 4x_1 + 5x_2 + 2x_3$

Subject to the constraints

$2x_1 + x_2 + x_3 \le 10$

$x_1 + 3x_2 + x_3 \le 12$

$x_1 + x_2 + x_3 = 6$

$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

4. Solve the L.P.P.

Max $z = 3x_1 + 2x_2$

Subject to the constraints

$2x_1 + x_2 \le 2$

$3x_1 + 4x_2 \ge 12$

$x_1, x_2 \ge 0$

5. Solve the L.P.P

Max $z = 5x_1 - 2x_2 + 3x_3$

Subject to the conditions

$2x_1 + 2x_2 - x_3 \ge 2$

$3x_1 - 4x_2 \le 3$

$2x_3 + 3x_3 \le 5,\ \ \ x_1, x_2, x_3 \ge 0.$

6. Show that the L.P.P.

Max $Z = 2x_1 + x_2$

Subject to the constraints

$x_1 - x_2 \le 10$

$2x_1 \le 40$

$x_1, x_2 \ge 0$ has a unbounded solution.

7. Solve the following problem using only one artificial variable.

$$\text{Maximize } Z = x_1 + 5x_2 + 3x_3$$

Subject to the constraints

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - x_2 = 4,$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

8. Solve the following problems using two-phase simplex method:

(i) Maximize $Z = x_1 + x_2 + x_3$

Subject to the constraints

$$x_1 - 3x_2 + 4x_3 = 5$$
$$x_1 - 2x_2 \leq 3$$
$$2x_2 - x_3 \geq 4$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(ii) Maximize $Z = 2x_1 + 5x_2$

Subject to the constraints

$$3x_1 + 2x_2 \geq 6$$
$$2x_1 + x_2 \leq 2$$
$$x_1 \geq 0, x_2 \geq 0.$$

(iii) Maximize $Z = 3x_1 + 2x_2 + 3x_3$

Subject to the constraints

$$2x_1 + x_2 + x_3 = 2$$
$$x_1 + 3x_2 + x_3 = 6$$
$$3x_1 + 4x_2 + 2x_3 = 8$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(iv) Maximize $Z = 2x_1 - 4x_2 + 3x_3$

Subject to the constraints

$$5x_1 - 6x_2 + 2x_3 \geq 5$$
$$-x_1 + 3x_2 + 5x_3 \geq 8$$
$$2x_1 + 5x_2 - 4x_3 \geq 4$$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ by using only two artificial variables.

(v) Maximize $Z = 2x_1 + 4x_2 + x_3$

Subject to the constraints

$$x_1 - 2x_2 - x_3 \leq 5$$
$$2x_1 - x_2 + 2x_3 = 2$$
$$-x_1 + 2x_2 + 2x_3 \geq 1$$
$$x_1 \geq 0, \ 1 \geq x_2 \geq 0 \text{ and } x_3 \geq 0.$$

9. For the following LP, show that the optimal solution is degenerate and that there exist alternative solutions that are all nonbasic.

$$\text{Maximize } Z = 3x_1 + x_2$$

$$\sum_{j=1}^{m} a_{ij} y_j \geq c_i, \quad 1 \leq i \leq n$$

$y_1, y_2, \ldots, y_m$ are unrestricted.

Note: The maximization (minimization) primal converts into minimization (maximization) in its dual.

1. The dual of the dual is the primal.
2. The cost vector associated with the variable x_i present in the primal becomes the right hand side of the i^{th} constraints in its dual.
3. The dual variables are unrestricted.

Example 1. Write down the dual of the following LPP

Maximize $Z = 5x_1 + 12x_2 + 4x_3$

Subject to the constraints

$$x_1 + 2x_2 + x_3 \leq 10$$
$$2x_1 - x_2 + 3x_3 = 8$$
$$x_1, x_2, x_3 \geq 0$$

Solution: The standard form of the LP is

Maximize $Z = 5x_1 + 12x_2 + 4x_3 + 0x_4$

Subject to the constraints

$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$2x_1 - x_2 + 3x_3 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Let y_1 and y_2 be the dual variables (since there are two equations in the constraints). Also the primal is to maximize (type 2). Therefore its dual is;

Minimize $W = 10y_1 + 8y_2$

Subject to the constraints

$$y_1 + 2y_2 \geq 5$$
$$2y_1 - y_2 \geq 12$$
$$y_1 + 3y_2 \geq 4$$
$$y_1 + 0y_2 \geq 0$$

y_1 and y_2 are unrestricted.

The last constraint indicates that only y_2 is unrestricted.

Example 2. Write the dual of the following linear programming problem:

$$\text{Maximize } Z = x_1 - x_2 + 3x_3$$

Subject to the constraints:

$$x_1 + x_2 - x_3 \le 3$$
$$2x_1 + 3x_2 - 4x_3 \ge 2$$
$$x_1 - x_3 \le -2$$
$$x_2 - 6x_3 = 4$$
$$x_1, x_3 \ge 0.$$

Solution: Since the variable x_2 is unrestricted, writing x_2 as $x_2^+ - x_2^-$ with $x_2^+ \ge 0, x_2^- \ge 0$, the given L.P.P. can be written in standard form as;

$$\text{Maximize } Z = x_1 - (x_2^+ - x_2^-) + 3x_3 + 0x_4 + 0x_5 + 0x_6.$$

Subject to the constraints:

$$x_1 + x_2^+ - x_2^- - x_3 + x_4 = 3$$
$$2x_1 + 3(x_2^+ - x_2^-) - 4x_3 - x_5 = 2$$
$$-x_1 + x_3 - x_6 = 2$$
$$x_2^+ - x_2^- - 6x_3 = 4$$
$$x_1, x_3, x_4, x_5, x_6, x_2^+, x_2^- \ge 0.$$

Let y_1, y_2, y_3 and y_4 be the dual variables (since there are 4 equations in the constraints). Also the primal is to maximize (type 2). Therefore its dual is;

$$\text{Minimize } W = 3y_1 + 2y_2 + 2y_3 + 4y_4$$

Subject to the constraints:

$$y_1 + 2y_2 - y_3 \ge 1$$
$$y_1 + 3y_2 + y_4 \ge -1$$
$$-y_1 - 3y_2 - y_4 \ge 1$$
$$-y_1 - 4y_2 + y_3 - 6y_4 \ge 3$$
$$y_1 + 0y_2 + 0y_3 + 0y_4 \ge 0$$
$$0y_1 - y_2 + 0y_3 + 0y_4 \ge 0$$
$$0y_1 + 0y_2 - y_3 + 0y_4 \ge 0$$

The last three equations implies that the variable y_4 is unrestricted and $y_2, y_3 \le 0$.

Example 3. Write the dual for the following primal problem

$$\textbf{Minimize } Z = 6x_1 + 3x_2$$

Subject to

$$6x_1 - 3x_2 + x_3 \geq 2$$
$$3x_1 + 4x_2 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Writing the given LP in the standard form we get

$$\text{Minimize } Z = 6x_1 + 3x_2 + 0x_4 + 0x_5.$$

Subject to

$$6x_1 - 3x_2 + x_3 - x_4 = 2$$
$$3x_1 + 4x_2 + x_3 - x_5 = 5$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Let y_1 and y_2 be the dual variables (since there are 2 equations in the constraints). Also the primal is to minimize (type 1). Therefore its dual is;

$$\text{Maximize } W = 2y_1 + 5y_2$$

Subject to the constraints:

$$6y_1 + 3y_2 \leq 6$$
$$-3y_1 + 4y_2 \leq 3$$
$$y_1 + y_2 \leq 0$$
$$-y_1 - y_2 \leq 0$$

y_1 and y_2 are unrestricted.

The last two equations implies that $y_1 + y_2 = 0$.

Example 4. Let the objective function of the L.P.P. is to maximize. Show that the general definition about the dual of the L.P.P. includes the following conditions;

1. **Any constraint with RHS is greater than the LHS corresponds to the non-negative dual variable corresponding to the inequality constraint and vice versa.**
2. **Any constraint with RHS is less than the LHS corresponds to the non-positive dual variable corresponding to the inequality constraint and vice versa.**
3. **Any equality constraint corresponds to the unrestricted dual variables and vice versa.**

Solution:

Consider an L.P.P. subject to the constraint $\sum_{i=1}^{k} a_{ij} x_i \le b_j \; 1 \le j \le m$, this inequality when writing in standard form, after the introduction of $n - k$ slack/surplus variables, reduces to $\sum_{i=1}^{n} a_{ij} x_i = b_j$, where $x_i \ge 0$, for all i, $1 \le i \le n$ ($n \ge k$). Let x_p be the slack variable corresponds to this j^{th} constraint in the primal. Then, as the objective function of the primal is to be maximize, the p^{th} constraint, $m < p \le n$, in the dual is;

$$0\,y_1 + 0\,y_2 + \cdots + 0\,y_{p-1} + y_p + 0\,y_{p+1} + \cdots + 0\,y_n \ge 0 \Rightarrow y_p \ge 0.$$

On the other hand if the constraint is $\sum_{i=1}^{k} a_{ij} x_i \ge b_j$, then the corresponding p^{th} constraint with respect to the surplus variable x_p is

$$0\,y_1 + 0\,y_2 + \cdots + 0\,y_{p-1} - y_p + 0\,y_{p+1} + \cdots + 0\,y_n \ge 0 \Rightarrow - y_p \ge 0 \Rightarrow y_p \le 0.$$

Finally if there exists any equation (equality constraint) then we are not getting any information about the corresponding variable y_p. Hence the dual variable y_p is unrestricted in this case. Converse part follows immediately as dual of the dual is the primal.

EXERCISES

I Write the dual for each of the following primal problems:

1. Max $Z = -5\,x_1 + 2\,x_2$

   Subject to

   $-x_1 + x_2 \le -2$

   $2\,x_1 + 3\,x_2 \le 5$

   $x_1, x_2 \ge 0$

2. Min $Z = 3\,x_1 + 4\,x_2 + 6\,x_3$

   Subject to

   $x_1 + x_2 \ge 10$

   $x_1 \ge 0, x_2 \le 0, x_3 \ge 0,$

3. Max $Z = x_1 + x_2$

   Subject to

   $2\,x_1 + x_2 = 5$

   $3\,x_1 - x_2 = 6$

   x_1 , x_2 unrestricted

4. Min $Z = 15\,x_1 + 12\,x_2$

   Subject to

   $x_1 + 2\,x_2 \ge 3$

   $2\,x_1 - 4\,x_2 \le 5$

   $x_1, x_2 \ge 0$

5. Min $Z = 20\,x_1 + 10\,x_2 + x_3$

   Subject to

   $3\,x_1 - 3\,x_2 + 5\,x_3 \le 50$

   $x_1 + x_3 \le 10$

   $x_1 - x_2 + 4\,x_3 \le 20$

   $x_1, x_2, x_3 \ge 0.$

II. Let the objective function of the L.P.P. is to minimize. Show that the general definition about the dual of the L.P.P. includes the following conditions:

(i) Any constraint with RHS is greater than the LHS corresponds to the non-positive dual variable corresponding to the inequality constraint and vice versa.

(ii) Any constraint with RHS is less than the LHS corresponds to the non-negative dual variable corresponding to the inequality constraint and vice versa.

(iii) Any equality constraint corresponds to the unrestricted dual variables and vice versa.

2.12 Relationship between the Optimal Primal and Dual Solutions

The primal and dual problems are so closely related that the optimal solution of one problem can be secured directly from the optimal simplex table of the other problem. The following are the properties on which our argument depends.

Property 1: At any simplex iteration of the primal or the dual

" The cost vector associated to the variable x_i in the objective function of the one problem is equal to the LHS minus RHS of the i^{th} constraint in the other problem"

Property 2: For any pair of feasible primal and dual solutions

" The maximum value of the objective function can not exceed the minimum value of the objective function ".

Property 1 can be used to determine the optimal solution of one problem directly from the optimal simplex table of the other. This result could be advantageous computationally if the computations associated with the solved problem are considerably less than those associated with the other. Property 2 shows that an optimal stage is reached, and then the value of the objective functions of the minimization problem is equal to the value of the maximization problem. If Z^* is the optimal value of the objective function associated with the dual, then the optimal value of the primal problem $Z = - Z^*$. Further if $z_j - c_j$ is the net evaluation in the corresponds to the starting basic (primal) variable, then the optimal solution/ value of the starting primal variable can be determined by the relation $x_i - c_i = -(z_i - c_j)$.

Example 5. Consider the following LPP

$$\text{Max } Z = 5x_1 + 2x_2 + 3x_3$$

Subject to

$$x_1 + 5x_2 + 2x_3 = 30$$
$$x_1 - 5x_2 - 6x_3 \leq 40$$
$$x_1, x_2, x_3 \geq 0.$$

The optimal solution yields the following objective equation

$$Z = 0x_1 - 23x_2 + 7x_3 + (5 + M)x_4 + 0x_5 = 150,$$

where artificial x_4 and slack x_5 are the starting basic variables. Write the associated dual problem and determine its optimal solution form the optimal Z-equation.

Solution: Given problem in standard form is

$$\text{Max } Z = 5x_1 + 2x_2 + 3x_3 + 0s_1$$

Subject to

$$x_1 + 5x_2 + 2x_3 = 30$$
$$x_1 - 5x_2 - 6x_3 + s_1 = 40$$
$$x_1, x_2, x_3, s_1 \geq 0.$$

Its dual is:

$$\text{Minimize } W = 30y_1 + 40y_2$$

Subject to

$$y_1 + y_2 \geq 5$$
$$5y_1 - 5y_2 \geq 2$$
$$2y_1 - 6y_2 \geq 3$$
$$y_1 \text{ is unrestricted, } y_2 \geq 0 \Rightarrow y_1 \geq -M$$

Since W is to minimize and $y_2 \geq 0$, $y_2 = 0$ (since W increases with increase in y_2).

But minimum value of W = maximum value of $Z = 150$

$$\Rightarrow 30y_1 + 40y_2 = 150 \Rightarrow 30(y_1) + 40(0) = 150$$
$$\Rightarrow y_1 = 5 \text{ (or Further by property 1, } y_1 - (-M) = 5 + M \Rightarrow y_1 = 5).$$

Hence 150 is the optimal value of the dual and $y_1 = 5$ and $y_2 = 0$ are the optimal solutions to the dual.

Example 6. Consider the following LPP

$$\textbf{Max } Z = 5x_1 + x_2 + 3x_3$$

Subject to

$$2x_1 + x_2 + x_3 = 3$$
$$x_1 - 2x_2 = -4$$
$$x_1, x_2, x_3 \geq 0$$

Write the associated dual problem. Given that the optimal basic variables are x_2 and x_3, determine the associated optimal dual solution.

Solution: Its dual is

$$\text{Minimize } W = 3y_1 - 4y_2$$

Subject to the constraints

$$2y_1 + y_2 \geq 5$$
$$y_1 - 2y_2 \geq 1$$
$$y_1 \geq 3$$
$$y_2 \text{ is unrestricted.}$$

Since x_2 and x_3 are the optimal feasible solutions, $x_1 = 0$. Substituting in the constraints of the primal (given) problem we get $x_2 = 2$ and $x_3 = 1$. The corresponding optimal value of $Z = 5(0) + (2) + 3\,(1) = 5$. Further W is to minimize and $y_1 \geq 3$, we have $y_1 = 3$ for the optimal value of W (since W increases with increase in y_1). Thus, min W = max $Z = 5$, we have

$$3\,y_1 - 4\,y_2 = 5 \Rightarrow 3\,(3) - 4\,y_2 = 5 \Rightarrow y_2 = 1.$$

Therefore, the associated dual solution is $y_1 = 3, y_2 = 1$.

Example 7. Consider the following LPP

$$\textbf{Max } Z = 2\,x_1 + 4\,x_2 + 4\,x_3 - 3\,x_4$$

Subject to the constraints

$$x_1 + x_2 + x_3 = 4$$

$$x_1 + 4\,x_2 + x_4 = 8$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Use the dual problem to verify that the basic solution (x_1, x_2) is not optimal.

Solution: The corresponding dual problem is

$$\text{Minimize } W = 4\,y_1 + 8\,y_2$$

Subject to the constraints

$$y_1 + y_2 \geq 2$$

$$y_1 + 4\,y_2 \geq 4$$

$$y_1 \geq 4$$

$$y_2 \geq -3$$

If x_1 and x_2 are optimal basic feasible solution. The $x_3 = x_4 = 0$. Hence by the constraints of the primal problem we get $x_1 = 8/3$ and $x_2 = 4/3$ and the corresponding optimal solution is $Z = 32/3$.

Further the value of W increases with $y_1 \geq 4$ and W is to minimize, thus optimal value attains at $y_1 = 4$.

Now Min W = max $Z = 32/3$ for the optimal solution $\Rightarrow 4\,y_1 + 8\,y_2 = 32/3 \Rightarrow 4\,(4) + 8\,y_2 = 32/3 \Rightarrow y_2 = -16/24$.

But as $y_1 = 4$, from second constraint of the dual problem we get $y_2 \geq 0$, a contradiction to the fact the $y_2 = -16/24$. Hence the pair $x_1,\ x_2$ cannot be basic feasible variables.

Example 8. Find the optimal value of the objective function for the following problem by only inspecting its dual (Do not solve the dual)

$$\textbf{Max } Z = 10x_1 + 4x_2 + 5x_3$$

Subject to

$$5x_1 - 7x_2 + 3x_3 \geq 50 \; ; \; x_1, x_2, x_3 \geq 0$$

Solution: The constrains of the dual problem are

$$5y_1 \geq 10, -7y_1 \geq 4, 3y_1 \geq 5 \text{ and} - y_1 \geq 0.$$

From the first constraint we get $y_1 \geq 2$ and the last one yields $y_1 \leq 0$. Hence there exists no value for y_1. Hence the dual has no feasible solution $\Rightarrow$ primal problem has an unbounded solution.

Example 9. Estimate the range for the optimal objective value for the following LPP

$$\textbf{Min } Z = 2x_1 + 5x_2$$

Subject to

$$x_1 - x_2 \leq 3 \; ; \; 3x_1 + 2x_2 \geq 5 \; ; \; x_1, x_2 \geq 0$$

Solution: Dual of the given problem is

$$\text{Max } W = 3y_1 + 5y_2$$

Subject to the constraints

$$y_1 + 3y_2 \leq 2$$

$$-y_1 + 2y_2 \leq 5$$

$$y_1 \leq 0 \text{ and } -y_2 \leq 0 \Rightarrow y_1 \leq 0 \text{ and } y_2 \geq 0.$$

Let $x_1{}^*$ and $x_2{}^*$ be optimal solution to Z, and, $y_1{}^*$ and $y_2{}^*$ be optimal solution to W. Then by third constraint $y_1 \leq 0$ of the dual implies that $y_1{}^* = 0$ (since W attains maximum when $y_1 = 0$).

Substituting this in remaining constraints of the dual, we get

$$0 + 3y_2{}^* \leq 2 \Rightarrow y_2{}^* = 2/3$$

$$0 + 2y_2{}^* \leq 5 \Rightarrow y_2{}^* \leq 5/2$$

$$-y_2{}^* \leq 0 \Rightarrow y_2{}^* \geq 0$$

These implies that $\quad 0 \leq y_2{}^* \leq 2/3 \quad$ ....(1)

Also at the optimal value, we have by the property 2

j^{th} coefficient of W = (LHS – RHS) of the constraint j of Z.

$$\Rightarrow 0 = x_1{}^* - x_2{}^* - 3 \qquad .....(2)$$

and $\quad y_2{}^* = 3x_1{}^* + 2x_2{}^* - 5 \qquad$ ......(3)

Substituting (3) in (1) we get

$$0 \le 3x_1^* + 2x_2^* - 5 \le 2/3$$

$$5 \le 3x_1^* + 2x_2^* \le 17/3$$

$$\Rightarrow \quad 15 \le 9x_1^* + 6x_2^* \le 17$$

$$\Rightarrow \quad 15 \le 9(3 + x_2^*) + 6x_2^*, \le 17 \quad \text{(from (2))}$$

$$\Rightarrow \quad 15 \le 12 + 15x_2^* \le 17$$

$$\Rightarrow \quad 3 \le 15x_2^* \le 5$$

$$\Rightarrow \quad 3/15 \le x_2^* \le 1/3$$

$$\Rightarrow \quad \boxed{1/5 \le x_2^* \le 1/3}$$

$$\Rightarrow \quad 1/5 \le x_1^* - 3 \le 1/3$$

$$\Rightarrow \quad (1/5) + 3 \le x_1^* \le (1/3) + 3$$

$$\Rightarrow \quad \boxed{16/5 \le x_1^* \le 10/3}$$

Further $Z = 2x_1 + 5x_2 \Rightarrow 2(16/5) + 5(1/5) \le Z \le 2(10/3)+5(1/3)$

$$\Rightarrow \quad \boxed{37/5 \le Z \le 25/3}$$

Example 10. Consider the L.P.P.

$$\textbf{Min } Z = 2x_1 + 5x_2$$

Subject to

$$x_1 - x_2 \le 3;\; 3x_1 + 2x_2 \ge 5;\; x_1, x_2 \ge 0$$

Let y_1 and y_2 be the dual variables. Determine whether the following pairs of primal dual solutions are optimal.

(i) $(x_1 = 1, x_2 = 3; y_1 = 1, y_2 = 4)$

(ii) $(x_1 = 0, x_2 = 3;\; y_1 = -5, y_2 = 0)$

Solution: Dual of the given problem is

Maximize $W = 3y_1 + 5y_2$

Subject to the constraints

$$y_1 + 3y_2 \le 2$$

$$-y_1 + 2y_2 \le 5$$

$$y_1 \le 0 \text{ and } -y_2 \le 0 \Rightarrow y_1 \le 0 \text{ and } y_2 \ge 0$$

(i) Given $x_1 = 1$ and $x_2 = 3$. We observe that these values satisfy the constraints of the given problem. Hence $x_1 = 1$ and $x_2 = 3$ are the basic feasible solutions. The corresponding value of Z is 18.

Given $y_1 = 1$ and $y_2 = 4$, which will not satisfy the constraint $y_1 \leq 0$ of the dual problem. Hence the given pair is not optimal solutions.

(ii) Given $x_1 = 0$ and $x_2 = 3$. We observe that these values satisfy the constraints of the given problem. Hence $x_1 = 1$ and $x_2 = 3$ are the basic feasible solutions. The corresponding value of Z is 15.

Given $y_1 = -5$ and $y_2 = 0$, which will satisfy all the constraints of the dual problem. The corresponding value of W is -15, which is not equal to the value of Z. Hence the given pair is a feasible solution to the dual and primal problems but not a optimal feasible soiutions.

Example 11. Applying principle of duality solve the following:

$$\text{Minimize } Z = 2x_1 + 2x_2$$

Subject to the constraints

$$2x_1 + 4x_2 \geq 1$$
$$x_1 + 2x_2 \geq 1$$
$$2x_1 + x_2 \geq 1$$
$$x_1, x_2 \geq 0.$$

Solution: The given problem in standard form is

$$\text{Minimize } Z = 2x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$2x_1 + 4x_2 - s_1 = 1$$
$$x_1 + 2x_2 - s_2 = 1$$
$$2x_1 + x_2 - s_3 = 1$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

The dual problem of the given primal is

$$\text{Maximum } W = y_1 + y_2 + y_3$$

Subject to the constraints

$$2y_1 + y_2 + 2y_3 \leq 2$$
$$4y_1 + 2y_2 + y_3 \leq 2$$
$$-y_1 \leq 0, -y_2 \leq 0 \text{ and } -y_3 \leq 0.$$
$$\Rightarrow \quad y_1 \geq 0, y_2 \geq 0 \text{ and } y_3 \geq 0.$$

Writing the dual problem in standard form after introducing the slack variables we get.

$$\text{Maximum } W = y_1 + y_2 + y_3 + 0\,s_1 + 0\,s_2 + 0\,s_3.$$

Subject to the constraints

$$2y_1 + y_2 + 2y_3 + s_1 = 2$$
$$4y_1 + 2y_2 + y_3 + s_2 = 2$$
$$y_1, y_2, y_3, s_1, s_2 \geq 0.$$

Initial iteration is

Simplex table – 1

		C	1	1	1	0	0	Ratio
C_B	y_B	x_B	y_1	y_2	y_3	s_1	s_2	x_{Bi}/y_{ij} $y_{ij>o}$
0	s_1	2	2	1	2	1	0	1
0	s_2	2	4	2	1	0	1	2
	z_j	0	0	0	0	0	0	
		$z_j - c_j$	-1	-1	-1	0	0	

y_3 should enter and s_1 should leave the basis.

Simplex table – 2

		C	1	1	1	0	0	Ratio
C_B	y_B	x_B	y_1	y_2	y_3	s_1	s_2	x_{Bi}/y_{ij}, $y_{ij>o}$
1	y_3	1	1	½	1	½	0	2
0	s_2	1	3	3/2	0	-½	1	2/3
	z_j	0	1	½	1	½	0	
		$z_j - c_j$	0	-½	0	½	0	

y_2 should enter and s_2 should leave the basis.

Simplex table – 3

		C	1	1	1	0	0	Ratio
C_B	y_B	x_B	y_1	y_2	y_3	s_1	s_2	x_{Bi}/y_{ij}, $y_{ij>o}$
1	y_3	2/3	0	0	1	2/3	-1/3	
1	y_2	2/3	2	1	0	-1/3	2/3	
	z_j	4/3	2	1	1	1/3	1/3	
		$z_j - c_j$	1	0	0	1/3	1/3	

Since net evaluation $z_j - c_j \geq 0$ for every column, the optimal solution is reached. The optimal value of the primal = optimal value of the dual = 4/3

To find optimal solution to the given primal

Net evaluation of the slack variables.............	:	1/3	1/3
Left hand side – right hand side of the constraints 4 and 5 of the given primal	:	$x_1 - 0$	$x_2 - 0$
Thus, from the equations (property 1)	:	$x_1 - 0 = 1/3$	$x_2 - 0 = 1/3$

We get $x_1 = 1/3$ and $x_2 = 1/3$ as the optimal solution to the given primal.

Example 12. Apply principle of duality to solve the L.P.P.

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to the constraints

$$2x_1 + x_2 \leq 3$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0.$$

Solution: Given problem in standard form is

$$\text{Max } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

Subject to the constraints

$$2x_1 + x_2 + s_1 = 3$$
$$3x_1 + 4x_2 - s_2 = 12$$
$$x_1 \geq 0, x_2 \geq 0$$

Its dual is;

$$\text{Min } Z = 3y_1 + 12y_2$$

Subject to the constraints

$$2y_1 + y_2 \geq 3$$
$$y_1 + 4y_2 \geq 2$$
$$y_1 \geq 0$$
$$-y_2 \geq 0 \quad \Rightarrow y_2 \leq 0$$

Writing the given Dual problem in Standard form we get

$$\text{Min } Z = 3y_1 - 12y_2 + 0s_1 + 0s_2 + Ma_1 + Ma_2.$$

Subject to the constraints

$$2y_1 - y_2 - s_1 + a_1 = 3$$
$$y_1 - 4y_2 - s_2 + a_2 = 2$$
$$y_1 \geq 0; \; y_2 \geq 0 \quad \text{(here we considered } y_2 \text{ as } -y_2)$$

Phase 1:
Simplex Table – 1

		C	0	0	0	0	-1	-1	Ratio
C_B	y_B	x_B	y_1	y_2	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}
-1	a_1	3	2	-1	-1	0	1	0	3/2
-1	a_2	2	1	-4	0	-1	0	1	2/1
	z_j	-5	-3	5	1	1	-1	-1	
	$z_j - c_j$		-3	5	1	1	0	0	

a_1 should leave the basis and y_1 should enter. Performing the operations $R_1 \leftarrow R_1/2$, $R_2 \leftarrow R_2 - R_1/2$, we get the next simplex table

Simplex table – 2

		C	0	0	0	0	-1	-1	Ratio
C_B	y_B	x_B	y_1	y_2	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}
0	y_1	3/2	1	- ½	- ½	0	½	0	--
-1	a_2	½	0	- 7/2	½	-1	- ½	1	1
	z_j	- ½	0	7/2	-½	1	½	-1	
		$z_j - c_j$	0	7/2	-½	1	3/2	0	
					↑				

From the table it is clear that the non-basic variable s_1 should enter the basis and the basic variable a_2 should leave the basis. Performing the operations $R_2 \leftarrow R_2/(1/2)$ and $R_1 \leftarrow R_1 + R_2$ we get

Simplex table – 3

		C	0	0	0	0	-1	-1	Ratio
C_B	y_B	x_B	y_1	y_2	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}
0	y_1	2	1	- 4	0	-1	0	1	
0	s_1	1	0	-7	1	-2	-1	2	
	z_j	**0**	0	0	0	0	0	0	
		$z_j - c_j$	0	0	0	0	1	1	

Since the net evaluation for each column is non-negative, we go for next phase

Phase – II
Simplex table – 4

		C	3	-12	0	0	Ratio
C_B	y_B	x_B	y_1	y_2	s_1	s_2	x_{Bi}/y_{ij}
3	y_1	2	1	- 4	0	-1	
0	s_1	1	0	-7	1	-2	
	z_j	6	3	-12	0	-3	
		$z_j - c_j$	0	0	0	-3	

Thus s_2 is the candidate to enter the basis. But $y_{ij} \leq 0$ for all i, in the column headed by s_2 there exists an unbounded solution to the dual $\Rightarrow$ the given primal has an infeasible solution.

Example 13. Using duality theory, solve the following LPP;

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to the constraints

$$x_1 + x_2 \geq 1$$

$$x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 10$$

$$x_2 \leq 3, \quad x_1, x_2 \geq 0$$

Solution: Given primal in standard form is

$$\text{Max } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3.$$

Subject to the constraints

$$-x_1 - x_2 + s_1 = -1$$

$$x_1 + x_2 + s_2 = 7$$

$$x_1 + 2x_2 + s_3 = 10$$

$$x_2 + s_4 = 3$$

$$x_1, x_2 \geq 0.$$

Its dual is

$$\text{Min } W = -y_1 + 7y_2 + 10y_3 + 3y_4$$

Subject to the constraints

$$-y_1 + y_2 + y_3 \geq 3$$

$$-y_1 + y_2 + 2y_3 + y_4 \geq 2$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0 \text{ and } y_4 \geq 0.$$

Introducing surplus and artificial variables, the first iterations is

Phase - I
Simplex table – 1

		C	0	0	0	0	0	0	-1	-1	Ratio
C_B	y_B	x_B	y_1	y_2	y_3	y_4	s_1	s_2	a_1	a_2	$x_{Bi}/y_{ij}, y_{ij} > 0$
-1	a_1	3	-1	1	1	0	-1	0	1	0	3
-1	a_2	2	-1	1	2	1	0	-1	0	1	1
	z_j	-5	2	-2	-3	-1	1	1	-1	-1	
		$z_j - c_j$	2	-2	-3	-1	1	1	0	0	

a_2 should leave the basis and y_3 should enter the basis.

Simplex table – 2

		C	0	0	0	0	0	0	1	1	Ratio
C_B	y_B	x_B	y_1	y_2	y_3	y_4	s_1	s_2	a_1	a_2	$x_{Bi}/y_{ij}, y_{ij} > 0$
-1	a_1	2	- ½	½	0	- ½	-1	½	1	- ½	4
0	y_3	1	- ½	½	1	½	0	- ½	0	½	2
	z_j	-2	-½	-½	0	½	1	- ½	-1	½	
		$z_j - c_j$	½	-½	0	½	1	- ½	0	3/2	

y_2 should enter and y_3 should leave the basis.

Simplex table – 3

		C	0	0	0	0	0	0	-1	-1	Ratio
C_B	y_B	x_B	y_1	y_2	y_3	y_4	s_1	s_2	a_1	a_2	$x_{Bi}/y_{ij}, y_{ij} > 0$
-1	a_1	1	0	0	-1	-1	-1	1	1	-1	1
0	y_2	2	-1	1	2	1	0	-1	0	1	--
	z_j	-1	0	0	1	1	1	-1	-1	1	
		$z_j - c_j$	0	0	1	1	1	-1	0	0	

s_2 should enter and a_1 should leave the basis.

Simplex table – 4

		C	0	0	0	0	0	0	-1	-1	Ratio
C_B	y_B	x_B	y_1	y_2	y_3	y_4	s_1	s_2	a_1	a_2	$x_{Bi}/y_{ij}, y_{ij} > 0$
0	s_2	1	0	0	-1	-1	-1	1	1	-1	
0	y_2	3	-1	1	1	0	0	0	1	0	
	z_j	**0**	0	0	0	0	0	0	0	0	
		$z_j - c_j$	0	0	0	0	0	0	1	1	

Phase 1 terminates here as $z_j - c_j \leq 0$ for all the columns and $Z = 0$.

Phase – II

Simplex table – 5

		C	-1	7	10	0	0	0	M	M	Ratio
C_B	y_B	x_B	y_1	y_2	y_3	y_4	s_1	s_2	a_1	a_2	$x_{Bi}/y_{ij}, y_{ij} > 0$
0	s_2	1	0	0	-1	-1	-1	1	1	-1	
7	y_2	3	-1	1	1	0	0	0	1	0	
	z_j	**21**	-7	7	7	0	0	0	0	0	
		$z_j - c_j$	-6	0	-3	0	0	0	$-M$	$-M$	

Since $z_j - c_j$ is non-positive for all the columns and the dual problem is minimize, the optimal solution is reached. The optimal value the primal problem is 21.

EXERCISES

1. Determine a feasible solution to the following set of inequalities by using the dual problem:

$$2x_1 + 3x_2 \leq 12$$
$$-3x_1 + 2x_2 \leq -4$$
$$3x_1 - 5x_2 \leq 2$$
$$x_1 \text{ unrestricted}$$
$$x_2 \geq 0$$

2. Consider the following LPP

$$\text{Max } Z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

Subject to the constraints

$x_1 + x_2 + x_3 = 4$

$x_1 + 4x_2 + x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

Given that $Z = 2x_1 + 0x_2 + 0x_3 + 3x_4 = 16$ is the objective equation in the optimal table. Determine the associated optimal dual solution.

3. Solve the following problem by dual simplex algorithm, and trace the path of the algorithm on the graphical solution space.

$$\text{Max } Z = 2x_1 + 4x_2$$

Subject to the constraints

$x_1 + x_2 = 1$

$-3x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0.$

2.13 Dual Simplex Method

In simplex method we observe that the net evaluations are independent x_B, the right hand side of the constraints. This implies that one or more of the values of x_b columns may be negative. In such cases, it is possible to find a starting basic (but not feasible) solution that is dual feasible. This idea is developed in the following algorithm:

Dual simplex algorithm:

Step 1. Write the given LPP in the standard form by introducing only slack variables (surplus variables may be converted into slack by multiplying throughout by –ve sign; right hand side may be negative). Convert the Minimization problem into Maximization (using the fact Min $Z = -$ Max $[-Z]$)

Step 2. Check the nature of x_{Bi}, for all i.

If $x_{Bi} \geq 0$, then apply simplex method to obtain optimum solution.

Else

If $z_j - c_j \geq 0$,

Select a vector y_k (row k) to leave from the basis such that

$$x_{bk} = \min_i \left\{x_{Bi} : x_{Bi} < 0\right\}.$$

Select a vector y_r (column r) to enter the basis such that

$$\left|\frac{z_r - c_r}{y_{kr}}\right| = \min_j \left\{\left|\frac{z_j - c_j}{y_{kj}}\right| : y_{kj} < 0\right\}$$, get the next simplex table, repeat step 2.

Else "the method is not applicable*" stop.

* However we can use artificial constraints in such cases, see example 3.

Example 1. Use dual simplex method to solve the LPP and trace the path of the algorithm on the graphical solution space.

$$\text{Minimize } Z = 4x_1 + 2x_2$$

Subject to the constraints

$$x_1 + x_2 = 1$$
$$3x_1 - x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Solution: The given problem contains an equation. We replace this equation by writing into two inequalities as

$$x_1 + x_2 \leq 1$$
$$x_1 + x_2 \geq 1 \text{ or } -x_1 - x_2 \leq -1$$

Also second constraint can be written as

$$-3x_1 + x_2 \leq -2$$

Now the given problem in standard form is

$$\text{Maximize } (-Z) = -4x_1 - 2x_2$$

Subject to the constraints

$$x_1 + x_2 + s_1 = 1$$
$$-x_1 - x_2 + s_2 = -1$$
$$-3x_1 + x_2 + s_3 = -2$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Simplex table – 1

		C	-4	-2	0	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3
0	s_1	1	1	1	1	0	0
0	s_2	−1	−1	−1	0	1	0
0	s_3 →	−2	−3	1	0	0	1
	z_j	0	0	0	0	0	0
		$z_j - c_j$	4	2	0	0	0
$k = 3$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		−4/3 ↑	-	-	-	-

Here most negative x_B is −2, hence the row 3 to be chosen first. In this row 3 only the entry − 3 is negative. The evaluation for this column is −4/3 (choose least negative if there are more than one such columns). From this it is clear that the variable s_3 should leave the basis and the variable x_1 should enter the basis. Performing the operations $R_3 \leftarrow R_3/(-3)$, $R_2 \leftarrow R_2 + R_3/(-3)$ and $R_1 \leftarrow R_1 - R_3/(-3)$ we get

Simplex table – 2

		C	– 4	– 2	0	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3
0	s_1	4/3	0	4/3	1	0	1/3
0	s_2 →	– 1/3	0	– 4/3	0	1	– 1/3
– 4	x_1	2/3	1	– 1/3	0	0	– 1/3
	z_j	0	– 4	4/3	0	0	4/3
		$z_j - c_j$	0	10/3	0	0	0
$k = 2$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		–	–10/4	–	–	0

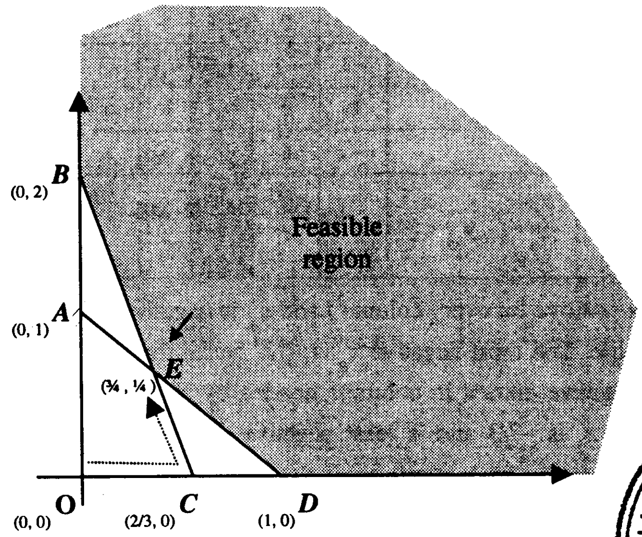
s_2 should leave the basis and x_2 should enter the basis. Performing the operations $R_2 \leftarrow -3R_2/4$, $R_1 \leftarrow R_1 + R_2$ and $R_3 \leftarrow R_3 - R_2/4$

Simplex table – 2

		C	– 4	– 2	0	0	0
y_B	C_B	x_B	x_1	x_2	s_1	s_2	s_3
s_1	0	1	0	0	1	1	0
x_2	– 2	¼	0	1	0	- ¾	¼
x_1	– 4	¾	1	0	0	– ¼	– ¼
	z_j	- 7/2	– 4	– 2	0	5/2	½
		$z_j - c_j$	0	0	0	5/2	½

Since the net evaluation is non-negative and $x_{Bi} \geq 0$ in all the rows, the optimum feasible solution is reached. Thus max $(-Z) = -7/2$. Thus the optimal value of the given problem is minimum $Z = -\max(-Z) = -(-7/2) = 7/2$ and the optimal solution is $x_1 = ¾$ and $x_2 = ¼$.

The solutions in each simplex table are; (0, 0), (2/3, 0) and (¾ , ¼). The feasible region for the above problem is shown below. The flow starts from O and ends at the optimal solution point E passing through the point C.



Example 2. Use dual simplex method to solve the following:

$$\textbf{Minimize } Z = 6x_1 + 7x_2 + 3x_3 + 5x_4$$

Subject to the constraints

$$5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12$$

$$x_2 - 5x_3 - 6x_4 \geq 10$$

$$2x_1 + 5x_2 + x_3 + x_4 \geq 8$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution: The given problem in standard form is

$$\text{Maximize } (-Z) = -6x_1 - 7x_2 - 3x_3 - 5x_4 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$-5x_1 - 6x_2 + 3x_3 - 4x_4 + s_1 = -12$$

$$-x_2 + 5x_3 + 6x_4 + s_2 = -10$$

$$-2x_1 - 5x_2 - x_3 - x_4 + s_3 = -8$$

$$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$$

Simplex table – 1

		C	−6	−7	−3	−5	0	0	0
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
0	s_1	−12	−5	−6	3	−4	1	0	0
0	s_2	−10	0	−1	5	6	0	1	0
0	s_3	−8	−2	−5	−1	−1	0	0	1
	z_j	0	0	0	0	0	0	0	0
		$z_j - c_j$	6	7	3	5	0	0	0
$k = 3$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		−6/5	−7/6 ↑	-	−5/4	-	-	-

Here $z_j - c_j$ is positive for every column j and x_{Bi} is negative for some i, hence dual simplex method is applicable. The most negative x_B is −12, hence the row 1 to be chosen first. In this row 1 contains negative entries in columns headed by x_1, x_2 and x_4. The evaluation for the column headed by x_2 is −7/3 and is least negative. Thus s_1 should leave the basis and the variable x_2 should enter the basis. Performing the operations $R_1 \leftarrow R_1/(-6)$, $R_2 \leftarrow R_2 + R_1/(-6)$ and $R_3 \leftarrow R_3 + 5\,R_1/(-6)$ we get

		C	−6	−7	−3	−5	0	0	0
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
−7	x_2	2	5/6	1	−½	2/3	−1/6	0	0
0	s_2	−8	5/6	0	9/2	20/3	−1/6	1	0
0	s_3	2	13/6	0	−7/2	−7/3	−5/6	0	1
	z_j	−14	−35/6	−7	7/2	−14/3	7/6	0	0
		$z_j - c_j$	1/6	0	13/2	1/3	7/6	0	0
$k = 3$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		-	·	-	-	−7	-	-

Thus s_2 should leave the basis and s_1 should enter the basis. Performing the operations $R_2 \leftarrow (-6)R_2$, $R_1 \leftarrow R_1 - R_2$ and $R_3 \leftarrow R_3 - 5\,R_2$, we get

		C	−6	−7	−3	−5	0	0	0
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
−7	x_2	10	0	1	−5	−6	0	−1	0
0	s_1	48	−5	0	−18	−40	1	−6	0
0	s_3	42	−2	0	−26	107/3	0	5	1
	z_j	−70	0	−7	35	42	0	7	0
		$z_j - c_j$	6	0	38	47	0	7	0

Since $z_j - c_j$ is positive for every column j and x_{Bi} is positive for each i, optimal feasible solution is reached. The optimal solution is $x_1 = 0$, $x_2 = 10$, $x_3 = 0$ and $x_4 = 0$. The optimal value of the objective function is Min $Z = -$ Max $[-Z] = -[-70] = 70$.

Example 3. Use dual simplex method to solve the following:

$$\textbf{Maximize } Z = x_1 - 3x_2$$

Subject to the constraints

$$x_1 - x_2 \le 2$$

$$x_1 + x_2 \ge 4$$

$$2x_1 - 2x_2 \ge 3$$

$$x_1, x_2 \ge 0.$$

Solution: The given problem in standard form is

$$\text{Maximize } Z = x_1 - 3x_2$$

Subject to the constraints

$$x_1 - x_2 + s_1 = 2$$

$$-x_1 - x_2 + s_2 = -4$$

$$-2x_1 + 2x_2 + s_3 = -3$$

$$x_1, x_2, s_1, s_2, s_3 \ge 0.$$

Simplex table – 1

		C	1	−3	0	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3
0	s_1	2	1	−1	1	0	0
0	s_2	−4	−1	−1	0	1	0
0	s_3	−3	−2	2	0	0	1
	z_j	**0**	0	0	0	0	0
		$z_j - c_j$	−1	3	0	0	0
$k = 3$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		↑				

The net-evaluation $z_j - c_j$ is negative for the first column, hence the dual simplex method is not applicable directly. For this, we add an artificial constraint whose LHS is the sum of variables (for which $z_j - c_j$ is negative) and less than a largest number M,

i.e. $x_1 \leq M$, where M is very large.

Now, the above simplex table, together with this artificial constraints yields,

		C	1	–3	0	0	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	2	1	–1	1	0	0	0
0	s_2	– 4	– 1	– 1	0	1	0	0
0	s_3	– 3	–2	2	0	0	1	0
0	s_4	M	1	0	0	0	0	1
	z_j	**0**	0	0	0	0	0	0
		$z_j - c_j$	–1	3	0	0	0	0
$k = 3$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		↑					

In this table we have to choose the new row (corresponds to the artificial constraint) as a pivotal row and the column 1 (corresponds to a variable in the artificial constraint). Applying this procedure to the above table, we get,

		C	1	–3	0	0	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	2	1	–1	1	0	0	0
0	s_2	– 4	–1	– 1	0	1	0	0
0	s_3	– 3	–2	2	0	0	1	0
0	s_4	M	1	0	0	0	0	1
	z_j	**0**	0	0	0	0	0	M
		$z_j - c_j$						
$k = 3$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		↑					

Performing the operations $R_1 \leftarrow R_1 - R_4$, $R_2 \leftarrow R_2 + R_4$ and $R_3 \leftarrow R_3 + 2R_4$, so as to enter the variable x_1 to the basis and to leave the variable s_4 from the basis we get,

Simplex table – 2

		C	1	-3	0	0	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	2-M	0	–1	1	0	0	-1
0	s_2	– 4+M	0	– 1	0	1	0	1
0	s_3	2M– 3	0	2	0	0	1	2
1	x_1	M	1	0	0	0	0	1
	z_j	M	1	0	0	0	0	M
		$z_j - c_j$	0	3	0	0	0	M
$k = 3$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		↑					

Since the net-evaluation is positive for all the columns. Choosing $M = 4$ (a minimum of possible largest number so as to get minimum negative basic variables x_{Bi}), we get

		C	1	-3	0	0	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1 →	–2	0	–1	1	0	0	-1
0	s_2	0	0	–1	0	1	0	1
0	s_3	5	0	2	0	0	1	2
1	x_1	4	1	0	0	0	0	1
	z_j	4	1	0	0	0	0	4
		$z_j - c_j$	0	3	0	0	0	4
$k = 3$	$\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0$		0	-3				-4

Here most negative x_B is –2, hence the row 1 to be chosen first. In this row 1 only the columns headed by x_2 and s_4 are negative. The evaluation for this column is – 3 and 0, the least negative is – 3. Thus the variable x_2 should enter basis and the variable s_1 should leave the basis. Performing the operations $R_1 \leftarrow R_1/(-1)$, $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 + 2R_1$, we get

Simplex table – 3

		C	1	-3	0	0	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	s_4	
-3	x_2 →	2	0	1	–1	0	0	1	2
0	s_2	2	0	0	–1	1	0	2	1
0	s_3	1	0	0	2	0	1	0	--
1	x_1	4	1	0	0	0	0	1	4
	z_j	–2	1	–3	3	0	0	–2	
		$z_j - c_j$	0	3	0	0	0	-3	

Here x_B is positive for all the rows, hence the dual simplex method terminates. Now we have to continue the simplex method corresponding to the maximization problem. But we observe that the net evaluation $z_j - c_j$ is negative for the column headed by s_4. Hence s_4 should enter the basis and s_2 should leave the basis. Performing the operations $R_2 \leftarrow R_2/(2)$, $R_1 \leftarrow R_1 - R_2/2$ and $R_3 \leftarrow R_3 - R_1/2$ we get,

		C	1	-3	0	0	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	s_4
–3	x_2	1	0	1	– ½	– ½	0	0
0	s_4	1	0	0	– ½	½	0	1
0	s_3	1	0	0	2	0	1	0
1	x_1	3	1	0	½	– ½	0	0
	z_j	**0**	1	-3	2	1	0	0
		$z_j - c_j$	0	0	2	1	0	0

The process terminates here as net-evaluation for every column is positive and $x_{Bi} \geq 0$. The optimal value is 0 and the optimal solution is $x_1 = 3$ and $x_2 = 1$.

EXERCISES

Using dual simplex method solve the following problems.

(i) Max $Z = x_1 + x_2$
Subject to
$2x_1 + 3x_2 \leq 12$
$-3x_1 + 2x_2 \leq -4$
$3x_1 - 5x_2 \leq 2$
x_1 unrestricted
$x_2 \geq 0$

(ii) Max $Z = 2x_1 + 4x_2 + 4x_3 - 3x_4$
Subject to the constraints
$x_1 + x_2 + x_3 = 4$
$x_1 + 4x_2 + x_4 = 8$
$x_1, x_2, x_3, x_4 \geq 0$

(v) Min $Z = -x_1 + x_2$
Subject to the constraints
$x_1 - 4x_2 \geq 5$
$x_1 - 3x_2 \leq 1$
$2x_1 - 5x_2 \geq 1$
$x_1, x_2 \geq 0.$

(iii) Max $Z = 2x_1 + 4x_2$
Subject to the constraints
$x_1 + x_2 = 1$
$-3x_1 + x_2 \geq 2$
$x_1, x_2 \geq 0.$

(iv) Max $Z = 2x_3$
Subject to the constraints
$-x_1 + 2x_2 - 2x_3 \geq 8$
$-x_1 + x_2 + x_3 \leq 4$
$2x_1 - x_2 + 4x_3 \leq 10$
$x_1, x_2, x_3 \geq 0.$

(vi) Min $Z = x_1 - x_2$
Subject to the constraints
$|x_1 - 4x_2| \geq 1$
$|x_1 - 3x_2| \leq 1$
$x_1, x_2 \geq 0.$

(vi) Max $Z = 2x_2$
Subject to the constraints
$-x_1 - 7x_2 + 3x_3 \geq 5$
$-x_1 - x_2 + x_3 \leq 1$
$3x_1 - 10x_2 + x_3 \leq 8$
$x_1, x_2, x_3 \geq 0.$

(viii) Min $Z = x_1 + x_2$
Subject to the constraints
$|x_1| - x_2 \geq 1$
$x_2 \geq 0$ and x_1 is unbounded.

(ix)) Min $Z = x_1 + x_2$
Subject to the constraints
$x_1 - 4x_2 = 5$
$x_1 - 3x_2 \leq 1$
$2x_1 - 5x_2 \geq 1$
$x_1, x_2 \geq 0$

3

TRANSPORTATION AND ASSIGNMENT PROBLEMS

3.1 Introduction

The transportation model is a special class of the linear programming and it deals with the transportation of a single product from several sources to several sinks. **The sources** are the places where the commodity is available and the **sinks** are the places where there is a demand. Sources are also called origins or supply or capacity centers and that the sinks are also called destinations or demand or requirement centers in different contests. The transportation network for the problem is a directed graph. Vertices represent the sources and sinks and the connecting roads are represented by edges. Further for each edge a weight is assigned, these weights are usually cost of transfer the goods on the road.

The objective of the transportation problem is to determine the amounts shipped from each source to each destination that minimize the total shipping cost while satisfying both the supply limits and the demand requirements.

The model assumes that the shipping cost on a given route is directly proportional to the number of units shipped on that route. Let x_{ij} and c_{ij} denote respectively the units of commodity shipped and cost of shipping one unit of goods from the source i to the sink j.

Let a_i and b_j denote the availability of the goods at the source i and the demand for the source at the sink j respectively. Then for a good transportation, the supply to the destination (sink) should be equal to the demand at that place and the supply of the good form each source equal to the availability at that place. Then the transportation problem is to determine a schedule so as to minimize the total cost satisfying the supply and demand conditions.

3.2 Formulation

Mathematically, the problem can be model as follows:

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad \text{(total cost)}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \,;\, i = 1, 2, 3, \ldots, m \qquad \text{(availability at source } i\text{)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \,;\, j = 1, 2, 3, \ldots, n \qquad \text{(demand at the sink } j\text{)}$$

and

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j \qquad \text{(non-negative supply)}$$

where m and n denote respectively the number of sources and sinks.

Together with this for a feasible solution to exist, it is necessary that total supply should be equal to the total requirements, *i.e.*

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \quad \text{called Rim Condition.}$$

Example 1: An automobile company has three distribution centers A, B and C. It manufactures cars at Puna and Delhi. The capacity of production at Puna and Delhi are respectively 1000 and 1500 cars for the next quarters. The quarterly demands at A, B and C are respectively 250, 1250 and 1000 cars. The cost of transport and the mileage chart are given in the following tables 1 and 2 respectively.

Table 1	mileage in kms		
↓Source Destination →	A	B	C
Puna	250	300	1200
Delhi	500	730	900

Table 2	Cost per km		
↓ Source Destination →	A	B	C
Puna	Rs. 5	Rs. 3	Rs 4
Delhi	Rs. 4	Rs. 3	Rs 5

Write down the LP model for the TP.

Solution: Let c_{ij} denote the cost of transfer the cars from the source i to destination j. Let x_{ij} denote the number of cars transferred from the source i to the destination j. Let source at Puna be 1 and that of Delhi 2. Let destinations A, B and C be respectively 1, 2 and 3. Then

$c_{11} = 5\times250 = 1250$, $c_{12} = 3\times300 = 900$, $c_{13} = 4\times1200 = 4800$, $c_{21} = 4\times500 = 2000$, $c_{22} = 3\times730 = 2190$ and $c_{23} = 5\times900 = 4500$.

Therefore the objective function is

$$\text{Minimize } z = \sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}\, x_{ij} = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23}$$

$$= 1250x_{11} + 900x_{12} + 4800x_{13} + 2000x_{21} + 2190x_{22} + 4500x_{23}$$

Subject to the constraints

$$x_{11} + x_{12} + x_{13} = 1000 \qquad \text{(capacity at Puna)}$$

$$x_{21} + x_{22} + x_{23} = 1500 \qquad \text{(capacity at Delhi)}$$

$$x_{11} + x_{21} = 250 \qquad \text{(demand at A)}$$

$$x_{12} + x_{22} = 1250 \qquad \text{(demand at B)}$$

$$x_{13} + x_{23} = 1000 \qquad \text{(demand at C)}$$

$$x_{ij} \geq 0 \text{ for all } i \in \{1,2\} \text{ and } j \in \{1,2,3\}$$

Note: The above TP we write in the following tabular form for the convenience, and is called transportation table.

↓Source Destination→	*A*	*B*	*C*	Availability
PUNA	x_{11} 1250	x_{12} 900	x_{13} 4800	1000
DELHI	x_{21} 2000	x_{22} 2190	x_{23} 4500	1500 ‖
Requirement	250	1250	1000 =	2500

The total demand in the above table is 2500 and the total supply is 2500. Hence the total demand is equal to the total supply. When the total supply does not equal to the total demand, the transportation model is said to be **unbalanced**. However such a model can be made balance by adding a dummy source or destination. The reason we are interested in balancing the transport model is that it allows the development of a solution algorithm that is based directly on the use of the transportation table.

3.3 Solution of Transportation Problem (TP)

The following are some basic steps involved in the computation for obtaining an optimum solution to a TP.

Step 1. Set up the transportation table with *m* rows representing the sources and *n* columns representing the sinks.

Step 2. Develop an initial feasible solution.

Step 3. Test whether the solution is an optimum one. If not, improve it further till the optimality is attained.

3.4 Finding Basic Feasible Solution

A general transportation model with *m* sources and *n* destinations has *m* + *n* constraints, one for each source and each destination. However, the transportation model is always balanced, hence one of these equations must be redundant. Thus the model has $m + n - 1$ independent constraint equations and hence the initial basic solution consists of $m + n - 1$ basic variables.

There are several methods available to obtain an initial basic feasible solution. We shall discuss the following three methods:

1. Northwest corner rule
2. Row minima and column minima method
3. Vogel's approximation method

The difference among the three methods is the "quality" of the starting basic solution they produce. In general Vogel method yields the best starting basic solution, and the northwest-corner method yields the worst. But the northwest-corner method involves the least computations.

3.5 The Northwest-Corner Rule

Step 1. Starting with the cell at the upper left (north-west) corner of the transportation matrix (table), we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied

$$i.e.\ x_{11} = \min(a_1, b_1)$$

Step 2. If $b_1 > a_1$, we move down vertically to the second row and make the second allocation of magnitude

$$x_{21} = \min(a_2, b_1 - x_{11}) \text{ in the cell } (2, 1).$$

If $b_1 < a_1$, we move right horizontally to the second column and make the second allocation of magnitude

$$x_{12} = \min(b_2, a_1 - x_{11}) \text{ in the cell } (1, 2).$$

Note: If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude

$$x_{12} = \min(b_1, a_1 - a_1) = 0 \text{ in the cell } (1, 2).$$

or $$x_{21} = \min(a_2, b_1 - b_1) = 0 \text{ in the cell } (2, 1).$$

Step 3. Go to step 1, repeat the process for the current cell.

Example 2. Obtain an initial basic feasible solution to the following transportation problem using northwest-corner rule

	D	*E*	*F*	*G*	*Available*
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Requirement	200	225	275	250	

Solution: The initial transportation table is

Source ↓ Destination →	D	E	F	G	Availability
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400 ‖
Requirement	200	225	275	250 =	950

From the above table we have the following

$c_{11} = 11, c_{12} = 13, c_{13} = 17, c_{14} = 14 \;;\; a_1 = 250$

$c_{21} = 16, c_{22} = 18, c_{23} = 14, c_{24} = 10 \;;\; a_2 = 300$

$c_{31} = 21, c_{32} = 24, c_{33} = 13, c_{34} = 10 \;;\; a_3 = 400$

$b_1 = 200, b_2 = 225, b_3 = 275, b_4 = 250$

Step 1. Starting with the upper left (north-west) corner cell (1, 1) of the transportation table, we allocate cell (1, 1) an allocation

$$x_{11} = \min(a_1, b_1) = \min(250, 200) = 200.$$

Step 2. We observe that $b_1 < a_1$ and hence we move right horizontally to the second column and make the second allocation of magnitude

$$x_{12} = \min(b_2, a_1 - x_{11}) = \min(225, 250 - 200) = 50$$

in the cell (1, 2).

Source ↓ Destination →	D	E	F	G	Availability
A	**200** → 11	**50** 13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400 ‖
Requirement	200	225	275	250 =	950

Step 3. We observe that no further allocation is possible for the first row as total availability at A is distributed to D and E. Since the requirement at E is 225 and 50 units are already supplied from A, we can now allocate a maximum of min (225 – 50, 300) to the cell (2, 2). Here we considered the minimum of requirement of E and availability at B to allocate to the cell (2, 2). Thus in general,

Step 1. an allotment to (2, 2) is

$$x_{22} = \min(b_2 - x_{12}, a_2) = \min(225 - 50, 300) = 175.$$

Step 2. Since the requirement in E is fulfilled, and (300-175) units are available at B that is less than the demand at B, this can be allotted to the cell (2,3) that is immediately right to the cell (2,2). Hence allocate

$$x_{23} = \min(a_2 - x_{22}, b_3) = \min(300 - 175, 275) = 125, \text{ to } (2,3)$$

Step 1. Similarly the cell (3, 3) can now be allotted a maximum allocation

$$x_{33} = \min(a_3 - x_{23}, b_3) = \min(275 - 125, 400) = 150$$

Step 2. Again moving right we allocate to the next cell (3,4) a maximum allocation

$$x_{34} = \min(a_3 - x_{33}, b_4) = \min(400 - 150, 250) = 250.$$

All the above steps shown in the following table; arrows indicate the allocation to the cells as per the steps.

Source ↓ Destination→	D	E	F	G	Availability
A	[200] 11	[50] 13	17	14	250
B	16	[175] 18	[125] 14	10	300
C	21	24	[150] 13	[250] 10	400
Requirement	200	225	275	250 =	950

Thus the starting basic solution is

$x_{11} = 200, x_{12} = 50, x_{13} = 0, x_{14} = 0, x_{21} = 0, x_{22} = 175, x_{23} = 125, x_{24} = 0, x_{31} = 0, x_{32} = 0,$
$x_{33} = 150$ and $x_{34} = 250$.

The associated cost of the schedule is

$$Z = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}\, x_{ij} = 11 \times 200 + 13 \times 50 + 18 \times 175 + 14 \times 125 + 13 \times 150 + 10 \times 250$$

$$= 12200$$

Example 3. Find a starting solution using northwest-corner method for the following problem given in tabulated form. Also find associated cost of the schedule.

Source ↓ Destination →	1	2	3	4	Availability
1	10	2	20	11	15
2	12	7	9	20	25
3	4	14	16	18	10
Requirement	5	15	15	15	

Solution: The arrows show the order in which the allocated amounts are generated

Source ↓ Destination →	1	2	3	4	Availability
1	[5] 10	[10] 2	20	11	15
2	12	[5] 7	[15] 9	[5] 20	25
3	4	14	16	[10] 18	10
Requirement	5	15	15	15	

The associated cost of the schedule is

$$z = 10 \times 5 + 2 \times 10 + 7 \times 5 + 9 \times 15 + 20 \times 5 + 18 \times 10 = 520$$

3. 6 Row Minima and Column Minima (Least-cost) Method

Step 1. Determine the smallest cost c_{ij} in the transportation table. Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j)

Step 2. If $x_{ij} = a_i$, cross off the i^{th} row of the transportation table and decrease b_j by a_i, Go to step 3.

If $x_{ij} = b_j$. cross off the j^{th} column of the transportation table and decrease a_i by b_j. Go to step 3.

If $x_{ij} = a_i = b_j$, cross off either the i^{th} row or the j^{th} column but not both.

Step 3. Repeat steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Example 4. Example: Find a starting solution using least cost method for the following problem given in tabulated form. Also find associated cost of the schedule.

Source ↓ Destination →	1	2	3	4	Availability
1	10	2	20	11	15
2	12	7	9	20	25
3	4	14	16	18	10
Requirement	5	15	15	15	

Solution: The arrows show the order in which the allocated amounts are generated.

Step 1. Min $\{c_{ij}\} = c_{12}$ (since the cell (1, 2) contains 2 which is the minimum of entries in the cells). Allocate $x_{12} = \min(a_1, b_2) = \min(15, 15) = 15$.

Step 2. $x_{12} = a_1 = b_2$, hence discard 1st row (or 2nd column). Reduce b_2 by 15.

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	15
2	12	7	9	20	25
3	4	14	16	18	10
Requirement	5	0	15	15	

Step 1. Smallest cost in the reduced matrix is 4, which is at the cell (3, 1). Hence assign $x_{31} = \min(a_3, b_1) = \min(10, 5) = 5$.

Step 2. $x_{31} = b_1 \neq a_3$, hence discard the column 1 and reduce a_3 by 5.

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	15
2	12	7	9	20	25
3	[5] 4	14	16	18	10-5=5
Requirement	5	15 − 15 = 0	15	15	

Step 1. Min $\{c_{ij}\} = c_{22}$ (since the cell (2, 2) contains 7 which is the minimum of entries in the cells). Allocate $x_{22} = \min(a_2, b_2) = \min(25, 0) = 0$.

Step 2. $x_{22} = b_2 \neq a_2$ hence discard the column 2. Reduce b_2 by 0.

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	15
2	12	[0] 7	9	20	25
3	[5] 4	14	16	18	5
Requirement	5	15 – 15 = 0	15	15	

Step 1. Min $\{c_{ij}\} = c_{23}$ (since the cell (2, 3) contains 9 which is the minimum of entries in the cells). Allocate $x_{23} = \min(a_2, b_3) = \min(25, 15) = 15$.

Step 2. $x_{23} = b_3 \neq a_2$ hence discard the column 3. Reduce a_2 by 15.

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	15
2	12	[0] 7	[15] 9	20	25-15=10
3	[5] 4	14	16	18	5
Requirement	5	0	15	15	

Step 1. Smallest cost in the reduced matrix is 18, which is at the cell (3, 4). Hence assign $x_{34} = \min(a_3, b_4) = \min(5, 15) = 5$.

Step 2. $x_{34} = a_3 \neq b_4$, hence discard the row 3 and reduce b_4 by 5.

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	15
2	12	[0] 7	[15] 9	20	25-15=10
3	[5] 4	14	16	[5] 18	5
Requirement	5	0	15	15	

Step 1. Smallest cost in the reduced matrix is 20, which is at the cell (2, 4). Hence assign $x_{24} = \min(a_2, b_4) = \min(10, 10) = 10$.

Step 2. $x_{34} = a_3 = b_4$, hence discard the row 2.

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	15
2	12	[0] 7	[15] 9	[10] 20	10
3	[5] 4	14	16	[5] 18	5
Requirement	5	0	15	10	

The process terminates here, the final table is

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	15
2	12	[0] 7	[15] 9	[10] 20	25
3	[5] 4	14	16	[5] 18	10
Requirement	5	15	15	15 =	50

The associated cost of the schedule is

$$z = 2 \times 15 + 4 \times 5 + 7 \times 0 + 9 \times 15 + 18 \times 5 + 20 \times 10 = 475.$$

Note: For the same problem by using northwest-corner method we got the initial cost of the schedule is 520. Hence the quality of the least-cost starting solution is better than that of the northwest-corner method.

3.7 Vogel's Approximation Method

Step 1. Calculate penalties by taking differences between the minimum and next to minimum unit transportation costs in each row and each column.

Step 2. Circle the largest Row difference or Column difference. In the event of tie, choose either. $- x_{11}$) in the cell (1, 2).

Step 3. Allocate as much as possible in the lowest cost cell of the row (or column) having a circled Row (or Column) difference.

Step 4. In case the allocation is made fully to a row (or column), ignore that row (or column) for further consideration by crossing it. If any row or column is left in the reduced matrix, then go to step 1. Otherwise Stop.

Example 5. Find a starting solution using Vogel's approximation method for the following problem given in tabulated form. Also find associated cost of the schedule.

Source ↓ Destination →	1	2	3	4	Availability
1	10	2	20	11	15
2	12	7	9	20	25
3	4	14	16	18	10
Requirement	5	15	15	15	

Solution:

Step 1. Minimum of the cost in the first row is 2 and the next minimum in the same row is 10. Thus row penalty (difference between these minimums) of the first row is 10 − 2 = 8. Minimum in the first column is 4 and the next minimum is 10, hence the penalty of the first column is 10 − 4 = 6. Similarly penalties for other rows and columns are computed in the following table

						Row penalty
	10	2	20	11	15	10-2=8
	12	7	9	20	25	9-7=2
	4	14	16	18	10	14-4=10
	5	15	15	15	50	
Column penalty	10-4 =6	7-2 =5	16-9 =7	18-11 =7		

Step 2. Largest penalty is 10 and the row 3 has assigned by this penalty.

Step 3. The third row contains the lowest cost 4 at the cell (3, 1). We allocate the maximum to the cell (3, 1) as

$$x_{31} = \min (a_3, b_1) = \min (10, 5) = 5$$

Step 4. Since the allocation to the first column is now complete, discarding the column and all the penalties we get,

						Row penalty
	10	2	20	11	15	
	12	7	9	20	25	
	[5] 4	14	16	18	10	
	5	15	15	15	50	
Column penalty						

Step 1. Minimum of the cost in the first row is 2 and the next minimum in the same row is 11. Thus row penalty (difference between these minimums) of the first row is 11 – 2 = 9. Minimum in the second column is 2 and the next minimum is 7, hence the penalty of the second column is 7 – 2 = 5. Similarly penalties for other rows and columns are computed in the following table

						Row penalty
	10	2	20	11	15	11-2=9
	12	7	9	20	25	9-7=2
	[5] 4	14	16	18	10	16-14=2
	5	15	15	15	50	
Column penalty		5	5	7		

Step 2. Largest penalty is 9 and the row 1 has assigned by this penalty.

Step 3. The row 1 contains the lowest cost 2 at the cell (1, 2). We allocate the maximum to the cell (1, 2) as

$$x_{12} = \min(a_1, b_2) = \min(15, 15) = 15$$

Step 4. Since the allocation to the column 2 and the row 1 is now complete, discarding the column 2, row 1 and reassigning the penalties, we get

	1	2	3	4		Row penalty
	10	[15] 2	20	11	15	
	12	7	9	20	25	20-9=11
	[5] 4	14	16	18	10	18-16=2
	5	15	15	15	50	
Column penalty			16-9 =7	20-18 =2		

Step 2. Largest penalty is 11 and the row 2 has assigned by this penalty.

Step 3. The row 2 contains the lowest cost 9 at the cell (2, 3). We allocate the maximum to the cell (2, 3) as

$$x_{23} = \min(a_2, b_3) = \min(25, 15) = 15$$

Step 4. Since the column 3 is now completely allocated, discarding the column 2 and assigning new penalties we get

						Row penalty
	10	[15] 2	20	11	15	
	12	7	[15] 9	20	25	20
	[5] 4	14	16	18	10	18
	5	15	15	15	50	
Column penalty				2		

Proceeding similarly for the new matrix we observe that the penalty 20 is maximum for the row 2, hence allocating a min (25-15, 15-0) = 10 to the cell (2, 4), the row 2 fulfill the requirement. Discarding the row 2,

						Row penalty
	10	[15] 2	20	11	15	
	12	7	[15] 9	[10] 20	25	
	[5] 4	14	16	18	10	(18)
	5	15	15	15	50	
Column penalty				18		

Choosing the final cell (3, 4) for the final allocation, the cell (3, 4) can be allocated with a maximum of min (15-10, 10-5) = 5.

						Row penalty
	10	[15] 2	20	11	15	
	12	7	[15] 9	[10] 20	25	
	[5] 4	14	16	[5] 18	10	(18)
	5	15	15	15	50	
Column penalty				18		

The flow of allocation is

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	**15**
2	12	7	[15] 9	[10] 20	**25**
3	[5] 4	14	16	[5] 18	**10**
Requirement	**5**	**15**	**15**	**15**	**50**

The associated objective value for this solution is

$$z = 4 \times 5 + 2 \times 15 + 9 \times 15 + 20 \times 10 + 18 \times 5 = 475.$$

This solution happens to have the same objective value as in the least-cost method. Usually, however, this method produces better starting solution for the TP.

EXERCISES

I. In each of the following cases determine whether a dummy source or a dummy destination must be added to balance the model

(i) Supply: $a_1 = 10, a_2 = 5, a_3 = 4, a_4 = 6$; Demand: $b_1 = 10, b_2 = 5, b_3 = 7, b_4 = 9$.

(ii) Supply: $a_1 = 30, a_2 = 40$; Demand: $b_1 = 25, b_2 = 30, b_3 = 10$.

(iii) Supply: $a_1 = 15, a_2 = 5, a_3 = 4, a_4 = 6$; Demand: $b_1 = 10, b_2 = 15, b_3 = 5$.

II. Find initial basic solution to the following TP using northwest-corner rule, Row minima and column minima, and Vogel's method. Also find associated cost of the schedule in each of the cases.

(i) Three electric power plants with capacities of 25, 40 and 30 million kWh. Supply electricity to three cities, the maximum demands at three cities are estimated at 30, 35 and 25 million kWh. The price per million kWh at the three cities are given in table:

		city 1	city 2	city 3
Plant	1	Rs. 600	Rs. 700	Rs. 400
	2	Rs. 320	Rs. 300	Rs. 350
	3	Rs. 500	Rs. 480	Rs. 450

During the month of August, there is a 20% increase in demand at each of the three cities, which can be met by purchasing the electricity from another network at a premium rate of Rs. 1000 per million kWh. The network is not linked to city 3, however. The utility company wishes to determine the most economical plan for the distribution and the purchase of additional energy.

(ii) Solve the problem number II (i) by assuming that there is a 10% power transmission loss through the network.

(iii) Three refineries with daily capacities of 6, 5 and 8 million gallons, respectively, supply three distribution areas with daily demands of 4, 8, and 7 million gallons respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is Rs. 10 per 1000 gallons per pipeline mile. The following table gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3.

		Distribution area 1	2	3
Plant	1	120	180	---
	2	300	100	80
	3	200	250	120

(iv) Automobiles are shipped from three distribution centers to five dealers. The shipped cost is based on the mileage between the sources and the destination, and is independent of whether the truck makes the trip with partial or full loads. The following table summarizes the mileage between the distribution centers and the dealers together with the monthly supply and demand figures given in number of automobiles. A full truckload includes 18 automobiles. The transportation cost for truck mail is Rs. 25.

		Dealer 1	2	3	4	5	Supply
Center	1	100	150	200	140	35	**400**
	2	50	70	60	65	80	**200**
	3	40	90	100	150	130	**150**
Demand		**100**	**200**	**150**	**160**	**140**	

(v) The demand of essential commodities for the next four months are 400, 300, 420 and 380 tones respectively. The supply capacities for the same months are 500, 600, 200 and 300 tones. The purchase price per tone varies from month to month is estimated at Rs. 100, Rs. 140, Rs. 120 and Rs. 150 respectively. Because the item is essential commodities, a current month's supply must be consumed within three months (including the current month). The storage cost per tone per month is Rs. 3. The nature of the item does not allow returning back.

(vi)

(a)

0	2	1	**6**
2	1	5	**9**
2	4	3	**5**
5	**5**	**10**	

(b)

0	4	2	**8**
2	3	4	**5**
1	2	0	**6**
7	**6**	**6**	

(c)

--	3	5	**4**
7	4	9	**7**
1	8	6	**19**
5	**6**	**19**	

(vii) In the unbalance transportation problem given in the following table, if a unit from a source is not shipped out (to any of the destinations), a storage cost is incurred at the rate of Rs. 5, Rs. 4 and Rs. 3 per unit for sources, 1, 2 and 3 respectively. If additionally all the supply at source 2 must be shipped out completely to make room for a new product,

1	2	1	**20**
3	4	5	**40**
2	3	3	**30**
30	**20**	**20**	

(viii) A departmental store wishes to stock the following quantities of a popular product in three types of container.

Container Type	1	2	3
Quantity	170	200	180

Tenders are submitted by four dealers who undertake to supply not more than the quantities shown below.

Dealer	1	2	3	4
Quantity	150	160	110	130

The store estimates that profit per unit bill vary with the dealer as shown below:

		Dealer 1	2	3	4
Container type	1	8	9	6	3
	2	6	11	5	10
	3	3	8	7	9

(ix) A company has received a contract to supply gravel for three new construction projects located in towns A, B and C. Construction engineers have estimated the required amounts of gravel which will be needed at these construction projects:

Project location	Weekly requirement (truck loads)
A	72
B	102
C	41

The company has 3 gravel pits located in town W, X and Y. The gravel required by the construction projects can be supplied by three pits. The amount of gravel which can be supplied by each pit is as follows:

Plant	W	X	Y
Amount available (truck loads)	76	82	77

The company has computed the delivery cost from each pit to each project site. These costs are shown in the following table.

		Project Location A	B	C
Pit	W	Rs. 4	Rs. 8	Rs. 8
	X	Rs. 16	Rs. 24	Rs. 16
	Y	Rs. 8	Rs. 16	Rs. 24

3.8 Finding the Optimal Solution using MODI Method

Consider a TP

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \,;\, i = 1, 2, 3, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \,;\, j = 1, 2, 3, \ldots, n$$

and $x_{ij} \geq 0$ for all i and j

Its dual is

$$\text{Maximize } z = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

Subject to the constraints

$$u_i + v_j \leq c_{ij}, \text{ for all } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

u_i and v_j unrestricted.

where u_i and v_j are dual variables of the constraint associated with the source i and destination j respectively.

Now from the prime-dual relationship, the objective function coefficients of the variable x_{ij} must be equal to the difference between the LHS – RHS of the corresponding dual constraint – that is, $u_i + v_j - c_{ij} = x_{ij}$. However for the basic variables we have $x_{ij} = 0 \Rightarrow u_i + v_j = c_{ij}$ for the basic variables.

Further, if there are $m + n - 1$ such equations $u_i + v_j = c_{ij}$ we can solve for u_i and v_j by taking $u_1 = 0$ or $v_1 = 0$ for each i and j. Once these multipliers u_i and v_j are computed, the entering variable is determined form all the nonbasic variables as the one having the largest positive net-evaluation $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$.

Basis on this MODI algorithm determines an optimum solution to the given TP starting from a initial solution to the TP that is determined using any one of the methods studied in the last section.

The various steps involved in solving any TP are;

Step 1. Find the initial basic feasible solution by using any of the three methods discussed above.

Step 2. Check the number of cells for which the allocation is made. If the number of such cells is less than $m + n - 1$, where m and n denote respectively the number of sources and sinks, there exists degeneracy and we introduce a very small positive assignment ε (≈ 0) in suitable independent positions, so that the number of occupied cells is exactly equal to $m + n - 1$.

Step 3. For each occupied cell (corresponds to basic variables) in the current solution, find u_i and v_j using the relation

$$c_{ij} = u_i + v_j, \text{ with } u_1 = 0 \text{ or } v_1 = 0$$

by entering successively the values of u_i and v_j in the transportation table.

Step 4. For each unoccupied cells (corresponds to non-basic variables) find the net evaluation $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$. Enter them in the lower right corners of the corresponding cells.

Step 5. Because the transportation model seeks to minimize cost, the entering variable to the basic solution should most positive net evaluation. Thus if all $z_{ij} - c_{ij} \leq 0$ for all the non-basic variables then the optimal feasible solution is reached. Otherwise choose the net evaluation that is most positive[⊗]. Starting form the non-basic cell having most positive net evaluation, add and subtract interchangeably a quantity $\theta = \min \{x_{ij} : (i, j)$ is a basic cell in the loop$\}$ to and from the transition cells of the loop in such a way that the rim requirements remain satisfied.

Step 6. Return to step 3, with the current assignment.

Example 1. Find the optimal solution to the following transportation problem:

	D_1	D_2	D_3	D_4	Available
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Required	22	35	25	41	123

The cell entries are unit transportation costs.

Solution:

To find initial basic feasible solution: by northwest-corner rule

1

[⊗] Most negative in the case of maximization problem.

Table – 0

[22] 23	[8] 27	16	18	30
12	[27] 17	[13] 20	51	40
22	28	[12] 12	[41] 32	53
22	35	25	41	123

The corresponding cost of the schedule is $Z = 22\times23+8\times27+27\times17+13\times20+12\times12+41\times32$
$= 2897$

Applying MODI method: **First iteration:** Number of basic cells $= m + n - 1 = 4 + 3 - 1$
$= 6$.

Basic variable	i	j	$c_{ij} = u_i + v_j$	Solution
x_{11}	1	1	$23 = 0 + v_1$	$v_1 = 23$
x_{12}	1	2	$27 = 0 + v_2$	$v_2 = 27$
x_{22}	2	2	$17 = u_2 + 27$	$u_2 = -10$
x_{23}	2	3	$20 = -10 + v_3$	$v_3 = 30$
x_{33}	3	3	$12 = u_3 + 30$	$u_3 = -18$
x_{34}	3	4	$32 = -18 + v_4$	$v_4 = 50$

The multipliers v_j we write on the top of column j and that of u_i on the left of the row i as:

$u_i \backslash v_j$	23	27	30	50	
0	[22] 23	[8] 27	16	18	30
-10	12	[27] 17	[13] 20	51	40
-18	22	28	[12] 12	[41] 32	53
	22	35	25	41	123

Now for the non-basic variables

Non-basic variable	i	j	$z_{ij} = u_i + v_j$	c_{ij}	Net evaluation $z_{ij} - c_{ij}$
x_{13}	1	3	$u_1 + v_3 = 0 + 30 = 30$	16	14
x_{14}	1	4	$u_1 + v_4 = 0 + 50 = 50$	18	32
x_{21}	2	1	$u_2 + v_1 = -10 + 23 = 13$	12	1
x_{24}	2	4	$u_2 + v_4 = -10 + 50 = 40$	51	-11
x_{31}	3	1	$u_3 + v_1 = -18 + 23 = 5$	22	-17
x_{32}	3	2	$u_3 + v_2 = -18 + 27 = 9$	28	-19